ON THE TEMPERATURE JUMP IN A RAREFIED GAS OVER A PERMEABLE SURFACE

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An expression is obtained for the temperature jump in a rarefied gas over a permeable surface on the basis of a numerical solution of the model kinetic equation in the Knudsen layer.

It is known that the expression for the temperature jump in a rarefied gas over an impermeable surface, obtained from the solution of the model kinetic equation in the Knudsen layer, differs from the Maxwell expression in the replacement of the factor $(2-\sigma)/\sigma$ by the factor $(2-k\sigma)/\sigma$, where $k = 0.827$ [1]:

$$\Delta T = \frac{75\pi}{128}\,\frac{2-0.827\sigma}{\sigma}\,l\left(\frac{dT}{dx}\right)_w. \tag{1}$$

In the present work the temperature jump is calculated from the solution of an analogous equation for a rarefied gas over a permeable wall

$$v_x\frac{\partial f}{\partial x} = \theta\,(f_M - f), \tag{2}$$

where f is the molecular velocity distribution function

$$f_M = \frac{n}{(2\pi RT)^{3/2}}\exp\left\{-\frac{(v_x-u)^2+v_y^2+v_z^2}{2RT}\right\}. \tag{3}$$

As in [1], let us assume that the gas density n and temperature T change only slightly in the transition domain so that they can be considered constant (and equal to $\bar{n}$ and $\bar{T}$, respectively) in the solution of (2) and only the gradients dn/dx and dT/dx depend on the coordinate x.

Let us represent the distribution function as follows:

$$f = f_0 + f_1, \tag{4}$$

where f_0 is some equilibrium distribution function close to f_M, and the correction f_1 is small compared to f_0.

Starting from the above-mentioned assumptions, let us write f_0 as

$$f_0 = \frac{\bar{n}}{(2\pi R\bar{T})^{3/2}}\exp\left\{-\frac{v^2}{2R\bar{T}}\right\}. \tag{5}$$

Let us assume that the mass flow rate of the gas u in the x direction is considerably less than the mean velocity of thermal motion $\bar{v}$ and is constant in the Knudsen layer ($u = u_0$).

By analogy with [1], substituting (3)-(5) into (2), we reduce the equation to the following:

$$v_x\frac{\partial f_1}{\partial x} + \theta f_1 = -v_x\frac{1}{\bar{n}}\frac{dn}{dx}(x)f_0 + v_x\frac{1}{\bar{T}}\left(\frac{3}{2}-\frac{v^2}{2R\bar{T}}\right)\frac{dT}{dx}(x)f_0 + \theta\frac{v_x u_0}{R\bar{T}}f_0. \tag{6}$$

Let us assume that the gas molecules reflected from the wall have a Maxwell distribution corresponding to the wall temperature T_w, i.e., the coefficient of accommodation σ equals one.

Taking into account that the ratios $(n_+ - n_0)/\bar{n}$ and $(T_w - T_0)/\bar{T}$ are small, let us write the boundary condition on the wall thus:

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$$f_1(0, \mathbf{v})_{v_x>0} = \frac{n_+ - n_0}{\bar{n}} f_0 - \frac{T_w - T_0}{\bar{T}} \left(\frac{3}{2} - \frac{v^2}{2R\bar{T}}\right) f_0. \quad (7)$$

On the basis of (6) and (7) we obtain for f_1

$$f_{1\,v_x>0} = \left[\frac{n_+ - n_0}{\bar{n}} \exp\left\{-\frac{\theta x}{v_x}\right\} - \frac{1}{\bar{n}} \int_0^x \frac{dn}{dx}(t) \exp\left\{\frac{\theta}{v_x}(t-x)\right\} dt\right] f_0 - \left(\frac{3}{2} - \frac{v^2}{2R\bar{T}}\right)$$

$$\times \left[\frac{T_w - T_0}{\bar{T}} \exp\left\{-\frac{\theta x}{v}\right\} - \frac{1}{\bar{T}} \int_0^x \frac{dT}{dx}(t) \exp\left\{\frac{\theta}{v_x}(t-x)\right\} dt\right] f_0 + \frac{u_0 v_x}{R\bar{T}} \left[1 - \exp\left\{-\frac{\theta x}{v_x}\right\}\right] f_0, \quad (8)$$

$$f_{1\,v_x<0} = \frac{f_0}{\bar{n}} \int_x^\infty \frac{dn}{dx}(t) \exp\left\{\frac{\theta}{v_x}(t-x)\right\} dt - \left(\frac{3}{2} - \frac{v^2}{2R\bar{T}}\right) \frac{f_0}{\bar{T}} \int_x^\infty \frac{dT}{dx}(t) \exp\left\{\frac{\theta}{v_x}(t-x)\right\} dt + \frac{u_0 v_x}{R\bar{T}} f_0.$$

The temperature jump is defined as the difference between the temperature T_0' (which is a linear extrapolation to the wall of the temperature curve in the domain bounding the layer near the wall) and the wall temperature T_w. Hence, we assume henceforth

$$\left(\frac{dT}{dx}\right)_w = \frac{dT}{dx}(x), \quad x \to \infty.$$

Using the mass and energy conservation laws, we find the unknown functions $(dn/dx)(x)$ and $(dT/dx)(x)$ from the expressions governing the mass and heat fluxes in a gas over a permeable wall:

$$\int_V v_x f d\mathbf{v} = \int_V v_x f_1 d\mathbf{v} = \bar{n} u_0,$$

$$\int_V \frac{1}{2} m v_x v^2 f d\mathbf{v} = \int_V \frac{1}{2} m v_x v^2 f_1 d\mathbf{v} = -\lambda \left(\frac{dT}{dx}\right)_w + \bar{n} m u_0 c_p \bar{T}. \quad (9)$$

The constants $(n_+ - n_0)/\bar{n}$ and $(T_w - T_0)/\bar{T}$ play the part of proper parameters.

Let us introduce the dimensionless variables

$$\xi = \frac{16}{15\pi^{1/2}} \frac{x}{l},$$

$$\mathbf{V} = \frac{\mathbf{v}}{(2R\bar{T})^{1/2}}, \quad U = \frac{u_0}{(2R\bar{T})^{1/2}},$$

$$\varphi(\xi) = \frac{1}{\bar{n}} \frac{dn}{d\xi}(\xi), \quad \chi(\xi) = \frac{1}{\bar{T}} \frac{dT}{d\xi}(\xi)$$

and let us define the functions

$$J_n(\xi) = \int_0^\infty V_x^n \exp\left\{-\frac{\xi}{V_x} - V_x^2\right\} dV_x,$$

$$L_{mn}(\xi) = \int_{V_x>0} V_x V^{2m} \left(\frac{3}{2} - V^2\right)^n \exp\left\{-\frac{\xi}{V_x} - V^2\right\} d\mathbf{V}.$$

Let us use the new variables

$$\varphi^*(\xi) = -\frac{\varphi(\xi)}{\chi_w}, \quad \chi^*(\xi) = \frac{\chi(\xi)}{\chi_w}$$

and the proper parameters

$$\mu_0 = \frac{n_+ - n_0}{\bar{n}\chi_w}, \quad \mu_1 = -\frac{T_w - T_0}{\bar{T}\chi_w}.$$

Substituting (8) into (9) and starting from the fact that

$$\theta = \frac{8}{15}\frac{\bar{v}}{l}, \quad \lambda = \frac{25\pi}{64}\rho\bar{v}lc_v,$$

$$\bar{v} = 2\left(\frac{2R\overline{T}}{\pi}\right)^{1/2}, \quad c_p = \frac{5}{2}R, \quad c_v = \frac{3}{2}R,$$

we obtain the following equations

$$\int_0^\infty L_{00}(|\xi-\tau|)\,\varphi^*(\tau)\,d\tau + \int_0^\infty L_{01}(|\xi-\tau|)\,\chi^*(\tau)\,d\tau = -\mu_0 L_{00}(\xi) - \mu_1 L_{01}(\xi) + 2\pi J_2(\xi)\frac{U}{\chi_w},$$
$$\int_0^\infty L_{10}(|\xi-\tau|)\,\varphi^*(\tau)\,d\tau + \int_0^\infty L_{11}(|\xi-\tau|)\,\chi^*(\tau)\,d\tau = -\mu_0 L_{10}(\xi) - \mu_1 L_{11}(\xi) + 2\pi\left[J_2(\xi)+J_4(\xi)\right]\frac{U}{\chi_w} - \frac{5}{4}\pi^{3/2}. \tag{10}$$

The system of integral equations (10) can be solved analytically, but we propose to use here a numerical solution which allows the determination of the temperature jump on a permeable surface to be pursued by a simpler method.

At the point $\xi = 0$, φ^*, χ^* take on infinite values. Hence, by analogy with [1], let us select a small positive number ε for which it can be assumed with sufficient accuracy that

$$\int_0^\varepsilon L_{mn}(|\xi-\tau|)\,\varphi^*(\tau)\,d\tau = L_{mn}(\xi)\int_0^\varepsilon \varphi^*(\tau)\,d\tau,$$

thereby introducing the new parameters

$$\mu_0^* = \mu_0 + \int_0^\varepsilon \varphi^*(\tau)\,d\tau, \quad \mu_1^* = \mu_1 + \int_0^\varepsilon \chi^*(\tau)\,d\tau.$$

They can be expressed from (10) by assuming $\xi = 0$. For μ_1^* we find

$$\mu_1^* = -\frac{2}{\pi^2}\left\{\frac{\pi}{2}\left[-\int_\varepsilon^\infty L_{10}(\tau)\,\varphi^*(\tau)\,d\tau - \int_\varepsilon^\infty L_{11}(\tau)\,\chi^*(\tau)\,d\tau - \frac{5}{4}\pi^{3/2}\right]\right.$$
$$\left. -\pi\left[-\int_\varepsilon^\infty L_{00}(\tau)\,\varphi^*(\tau)\,d\tau - \int_\varepsilon^\infty L_{01}(\tau)\,\chi^*(\tau)\,d\tau\right]\right\} - \frac{\pi^{1/2}}{4}\frac{U}{\chi_w}. \tag{11}$$

Writing φ^* and χ^* as

$$\varphi^* = \varphi_n^* + \Delta\varphi_n^*, \quad \chi^* = \chi_n^* + \Delta\chi_n^*$$

and substituting the expressions for μ_0^* and μ_1^* into (10), we obtain a system of equations to determine $\Delta\varphi_n^*$ and $\Delta\chi_n^*$:

$$\Delta\varphi_n^*(\xi)\,K_{00}(\xi) + \Delta\chi_n^*(\xi)\,K_{01}(\xi) = f_0(\xi) - \int_\varepsilon^\infty L_{00}^*(\xi,\tau)\left[\varphi_n^*(\tau)-1\right]d\tau - \int_\varepsilon^\infty L_{01}^*(\xi,\tau)\left[\chi_n^*(\tau)-1\right]d\tau + P_0(\xi)\frac{U}{\chi_w},$$
$$\Delta\varphi_n^*(\xi)\,K_{10}(\xi) + \Delta\chi_n^*(\xi)\,K_{11}(\xi) = f_1(\xi) - \int_\varepsilon^\infty L_{10}^*(\xi,\tau)\left[\varphi_n^*(\tau)-1\right]d\tau - \int_\varepsilon^\infty L_{11}(\xi,\tau)\left[\chi_n^*(\tau)-1\right]d\tau + P_1(\xi)\frac{U}{\chi_w}, \tag{12}$$

where

$$L_{mn}^*(\xi,\tau) = L_{mn}(|\xi-\tau|) - L_{m0}(\xi)\left[\frac{3}{\pi}L_{0n}(\tau) - \frac{1}{2\pi}L_{1n}(\tau)\right] - L_{m1}(\xi)\left[\frac{2}{\pi}L_{0n}(\tau) - \frac{1}{\pi}L_{1n}(\tau)\right];$$

$$K_{mn}(\xi) = \int_\varepsilon^\infty L_{mn}^*(\xi,\tau)\,d\tau;$$

$$f_m(\xi) = L_m(\xi) - K_{m0}(\xi) - K_{m1}(\xi);$$

$$P_0(\xi)=2\pi J_2(\xi)-\frac{7}{8}\pi^{1/2}L_{00}(\xi)+\frac{\pi^{1/2}}{4}L_{01}(\xi);$$

$$P_1(\xi)=2\pi\left[J_2(\xi)+J_4(\xi)\right]-\frac{7}{8}\pi^{1/2}L_{10}(\xi)+\frac{\pi^{1/2}}{4}L_{11}(\xi).$$

The system (12) was solved by iteration on the Minsk-22 electronic computer for $U/\chi_w = 0; 1$.

The temperature distribution and temperature jump are defined as:

$$T(\xi)=T_0+\int_0^\xi \chi^*(\tau)\,d\tau\left(\frac{dT}{d\xi}\right)_w,$$

$$\Delta T=T_0'-T_w=T_0-T_w+\int_0^\infty[\chi^*(\tau)-1]\,d\tau\left(\frac{dT}{d\xi}\right)_w. \tag{13}$$

The temperature T_0 can be represented as

$$T_0=T_w+\mu_1\left(\frac{dT}{d\xi}\right)_w. \tag{14}$$

Substituting (14) into (13), we obtain

$$\Delta T=\left\{\mu_1+\int_0^\infty[\chi^*(\tau)-1]\,d\tau\right\}\left(\frac{dT}{d\xi}\right)_w=\left\{\mu_1^*-\varepsilon+\int_\varepsilon^\infty[\chi^*(\tau)-1]\,d\tau\right\}\left(\frac{dT}{d\xi}\right)_w. \tag{15}$$

Let us write (11) as

$$\mu_1^*=\mu_2^*-\frac{\pi^{1/2}}{4}\,\frac{U}{\chi_w}. \tag{16}$$

From (15) and (16), we find

$$\Delta T=\frac{75\pi}{128}(2-k)\,l\left(\frac{dT}{dx}\right)_w-\frac{\pi^{1/2}\overline{T}}{4}U, \tag{17}$$

where k is a coefficient the value of which is defined by the relationship

$$k=2-\frac{8}{5\pi^{1/2}}\left\{\mu_2^*-\varepsilon+\int_\varepsilon^\infty[\chi^*(\tau)-1]\,d\tau\right\}.$$

It should be noted that the expression for the temperature jump on a permeable surface has been derived in a thirteen-moment approximation in [2, 3]. It is seen from a comparison of (17) and the results in these papers that the first member in (17) differs from the analogous terms in [2, 3] by the presence of a factor 2−k but the members containing the velocity U agree.

In general the coefficient k depends on U/χ_w. However, computations have shown that this dependence is quite weak (for $U/\chi_w = 0$ k = 0.826, and for $U/\chi_w = 1$ k = 0.830), and it can be neglected in practice. For U = 0, k agrees well with the result presented in [1].

The temperature jump can also be represented as

$$\Delta T=\frac{75\pi}{128}(2-k_{\text{eff}})\,l\left(\frac{dT}{dx}\right)_w,$$

where

$$k_{\text{eff}}=k+0.4\,\frac{U}{\chi_w}.$$

The difference between k and k_{eff} characterizes the contribution of the term containing the velocity to the magnitude of the temperature jump.

NOTATION

θ is the collision frequency;
x is the coordinate along the normal to the wall;
v is the velocity of molecule in a fixed coordinate system;
m is the mass of molecule;
l is the mean free path;
c_p, c_v are the specific heats of the gas at constant pressure and constant volume, respectively;
ρ, λ are the gas density and coefficient of heat conduction;
R is the gas constant;
n_0, T_0 are the number of molecules per unit volume and gas temperature at the wall, respectively;
n_+ is the number of molecules per unit volume in the stream of molecules reflected at the wall.

LITERATURE CITED

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